ORC 67-4
JANUARY 1967

THE DISTANCE FUNCTION OF A PRODUCTION STRUCTURE

by

Ronald W. Shephard

OPERATIONS RESEARCH CENTER

COLLEGE OF ENGINEERING

UNIVERSITY OF CALIFORNIA - BERKELEY

THE DISTANCE FUNCTION OF A PRODUCTION STRUCTURE

by

Ronald W. Shephard
Department of Industrial Engineering
and Operations Research
University of California, Berkeley

January 1967 ORC 67-4

This research was supported in part by the National Science Foundation Grant No. GP-4593 and the Office of Naval Research Contract No. Nonr - 222(83) with the the University of California. Reproduction in whole or in part is permitted for any purpose of the United States Government.

THE DISTANCE FUNCTION OF A PRODUCTION STRUCTURE

by

Ronald W. Shephard

1. DEFINITION OF THE DISTANCE FUNCTION

Consider a production structure with a production function $\Phi(x)$ defined on $D = \{x \mid x \geqq 0 , x \in R^n\}$ having the properties:

A.1 $\Phi(0) = 0$.

A.2 $\Phi(x)$ is finite for x finite.

A.3 $\Phi(x') \geqq \Phi(x)$ for $x' \geqq x$.

A.4* If $x > 0$, or $x \geq 0$ and $\Phi(\lambda x) > 0$ for some $\lambda > 0$, $\Phi(\lambda x) \to \infty$ as $\lambda \to \infty$.

A.5 $\Phi(x)$ is upper semi-continuous on D .

A.6 $\Phi(x)$ is quasi-concave on D .

As shown in [3], the level sets $L(u) = \{x \mid \Phi(x) \geqq u , x \in D\}$ have the properties:

P.1 $L(0) = D$, $0 \notin L(u)$ for $u > 0$.

P.2 If $x \in L(u)$ and $x' \geqq x$, then $x' \in L(u)$.

P.3 If $x > 0$, or $x \geq 0$ and $(\lambda x) \in L(u)$ for some $\lambda > 0$ and $u > 0$, the ray $\{\lambda x \mid \lambda \geqq 0\}$ intersects all level sets $L(u)$, $u \in [0,\infty)$.

P.4 $L(u_2) \subset L(u_1)$ if $u_2 \geqq u_1$.

P.5 $\bigcap_{0 \leqq u < u_0} L(u) = L(u_0)$ for any $u_0 > 0$.

---

* $x \geq y \Rightarrow x \geqq y$ but $x \neq y$.

P.6 $\bigcap_{u \varepsilon [0,\infty)} L(u)$ is empty.

P.7 $L(u)$ is closed for any $u \varepsilon [0,\infty)$.

P.8 $L(u)$ is convex for any $u \varepsilon [0,\infty)$.

Starting with the production possibility sets $L(u)$, i.e., for any output rate $u \varepsilon [0,\infty)$ the subset of D for which output rate is equal to or greater than u , the properties P.1, ... P.8 are technological assumptions for a general structure of production, and a unique function $\Phi(x) = \underset{L(u) \supset x}{\text{Max}}\, u$ may be defined on the sets $L(u)$ which has the properties A.1, ... A.6. (See [3].) This definition leads to the classical production function.

For reasons of studying the relationships between cost functions and production functions, it is convenient to define another function on the level sets $L(u)$ which serves to give an alternative definition of a production function. This function is an adaption for the production possibility sets $L(u)$ of the Minkowski distance function for convex bodies* [2].

The factor input domain D may be partitioned into mutually exclusive and exhaustive subsets as follows:

$$\{0\}$$

$$D_1 = \{x \mid x > 0\}$$

$$D_2 = \left\{x \mid x \geq 0 \ , \ \prod_1^n x_i = 0\right\}$$

where D_1 is the set of interior points of D and $\{0\} \cup D_2$ comprises the boundary of D . Further D_2 may be partitioned into

*A bounded convex closed set in R^n is a convex body.

$$D_1' = \{x \mid x \in D_2 \text{ , } (\lambda x) \in L(u) \text{ for some } u > 0 \text{ and } \lambda > 0\}$$

$$D_2'' = \{x \mid x \in D_2 \text{ , } (\lambda x) \notin L(u) \text{ for all } u > 0 \text{ , } \lambda > 0\} \text{ .}$$

All points of D_2 belong to either D_2' or D_2'', since if $(\lambda x) \in L(u)$ for some $u > 0$ and $\lambda > 0$, then for each $u > 0$ there is a $\lambda > 0$ such that $(\lambda x) \in L(u)$ by virtue of the property P.3. Thus

$$D = \{0\} \cup D_1 \cup D_2' \cup D_2''$$

where $\{0\}$, D_1, D_2', D_2'' are mutually exclusive.

A nonnegative distance function $\Psi(u,x)$ is defined on D for the level sets $L(u)$, $u \in [0,\infty)$ by

$$\Psi(u,x) = \begin{cases} \dfrac{||x||}{||\xi||} & \text{for } x \in D_1 \cup D_2' \\ 0 & \text{for } x \in \{0\} \cup D_2'' \end{cases} \tag{1}$$

where

$$\xi = \lambda_o x \text{ and } \lambda_o = \text{Min } \lambda \text{ for } \lambda x \in L(u) \text{ .} \tag{2}$$

If $x \in D_1 \cup D_2'$, it follows from the property P.3 that the ray $\{\lambda x \mid \lambda \geqq 0\}$ intersects all production possibility sets $L(u)$, $u \in [0,\infty)$ --- and for any $u > 0$ the intersection ξ exists with $||\xi|| > 0$, since $L(u)$ is closed (p.7) and $\xi \neq 0$ by P.1. Note that if $x \in D_2'$ the ray $\{\lambda x \mid \lambda \geqq 0\}$ intersects the sets $L(u)$ only on their boundaries, but the definition (2) implies that the intersection ξ with smallest norm is used in (1) (see Figure 1).

When $x \in \{0\} \cup D_2''$, the ray $\{\lambda x \mid \lambda \geqq 0\}$ is either not defined for $x = 0$ or fails to intersect each set $L(u)$, $u > 0$; however, zero is a natural value for $\Psi(u,x)$ in both cases, which may be seen by considering the perturbed points $(x'' + \Delta)$, $(0 + \Delta)$ where $\Delta \in D_1 \cup D_2'$, and taking the limit as $\Delta \to 0$ through points of $D_1 \cup D_2'$. In fact

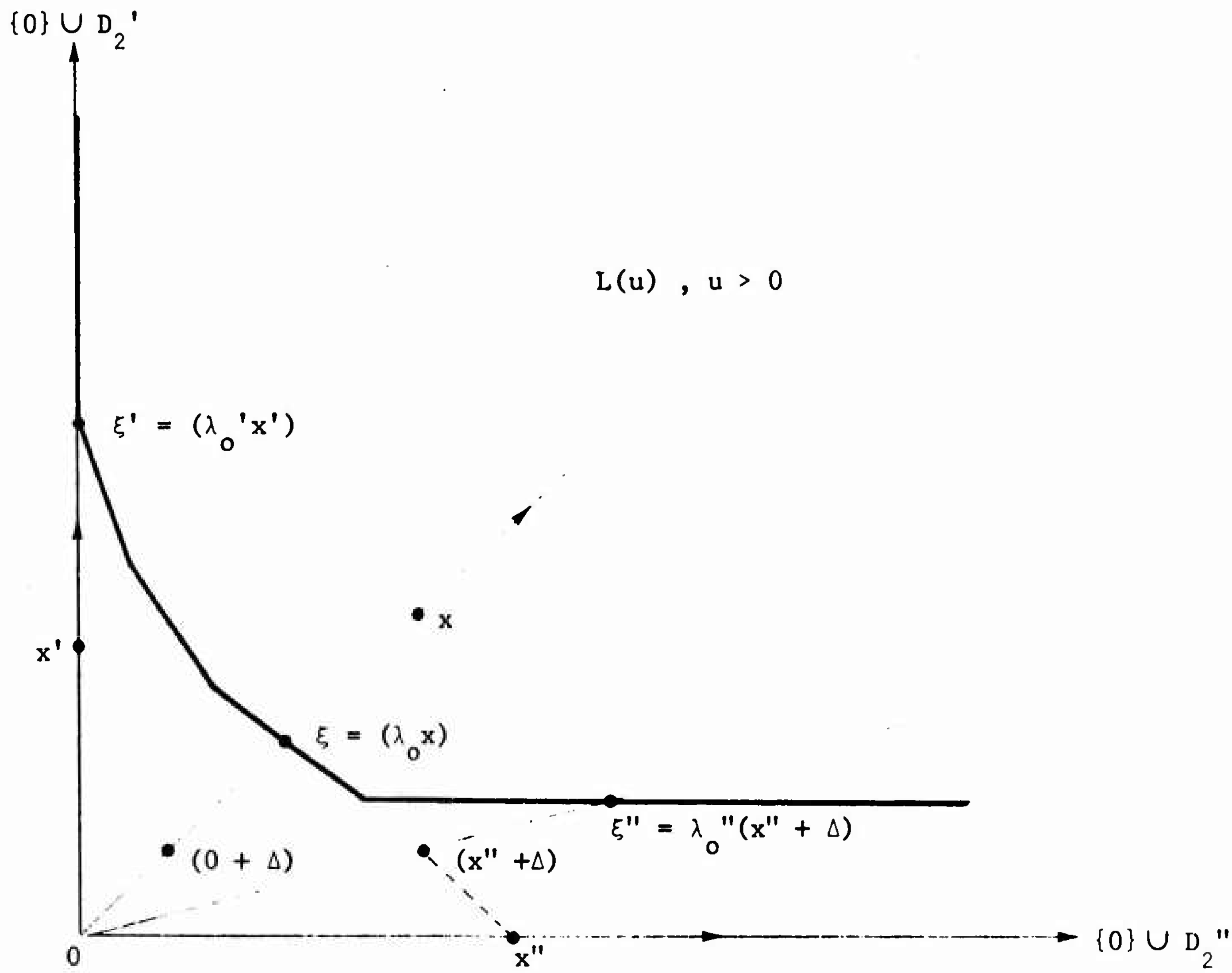

FIGURE 1: INTERSECTIONS OF A LEVEL SET $L(u)$ BY RAYS FROM THE ORIGIN

$$\Psi(u\ ,\ x'' + \Delta) = \frac{||x'' + \Delta||}{||\lambda_o''(x'' + \Delta)||}\ ,\ x'' \in D_2''$$

and, as $\Delta \to 0$, $||\lambda_o''(x'' + \Delta)|| \to \infty$ so that $\Psi(u\ ,\ x'' + \Delta) \to 0$, because if the $\lim_{\Delta \to 0} ||\lambda_o''(x'' + \Delta)||$ were finite the point $(\lambda_o''x'') \in L(u)$ due to the closure of $L(u)$, contrary to the definition of the set D_2'' . Also, for $x = 0$,

$$\Psi(u\ ,\ 0 + \Delta) = \frac{||0 + \Delta||}{||\lambda_o(0 + \Delta)||}$$

and, as $\Delta \to 0$, $||0 + \Delta|| \to 0$ while $||\lambda_o(0 + \Delta)||$ is bounded away from zero since $\xi \neq 0$ by P.1.

The foregoing definition of a distance function for each set $L(u)$, $u > 0$, does not have the properties of the Minkowski distance function, because 0 is not an interior point of the level set $L(u)$, the set $L(u)$ is not bounded and the distance function is super-additive. However, it is suitable for our purposes, since we are not concerned whether $\Psi(u,x)$ has the sub-additive property of the Minkowski distance function.

With the definition of $\Psi(u,x)$ given by (1) and (2), the production possibility sets $L(u)$ may be expressed by:

<u>Proposition 1:</u> For any $u \in [0,\infty)$

$$L(u) = \{x \mid \Psi(u,x) \geqq 1\ ,\ x \in R^n\}\ ,\ u > 0 \qquad (3)$$
$$L(0) = D\ .$$

Clearly, if $x \in L(u)$ then $\xi \leqq x$ and $\left|\left|\frac{x}{\xi}\right|\right| \geqq 1$. Also, if $x \notin L(u)$ and $x \in D_1 \cup D_2'$, then $x < \xi$ and $\left|\left|\frac{x}{\xi}\right|\right| < 1$, and when $x \in \{0\} \cup D_2''$, $\Psi(u,x) = 0 < 1$. Thus , $x \in L(u)$ for $u > 0$ if and only if $\Psi(u,x) \geqq 1$. For $u = 0$, we merely use property P.1 to define $L(0)$.

The boundary points of the sets $L(u)$ are conveniently expressed by:

<u>Proposition 2:</u> The boundary points of the production possibility sets $L(u)$, $u \in [0,\infty)$ are characterized by:

For $u > 0$, $x \in$ Boundary $L(u)$ if and only if $\Psi(u,x) = 1$ when $x \in D_1$ and $\Psi(u,x) \geqq 1$ when $x \in D_2'$. For $u = 0$, $\{0\} \cup D_2$ is the boundary of $L(0)$.

If $u > 0$, it is clear from the definition of D_1 and D_2' and property P.1, that the boundary points of a set $L(u)$, $u > 0$ belong to either D_1 or D_2'. When $x \in D_1$ and $x \in$ Boundary $L(u)$, then $\xi = x$ and $\Psi(u,\xi) = 1$; also $\Psi(u,x) = 1$ implies $\xi = x$. However, when $x \in D_2'$, points λx where $\lambda \geqq \lambda_o$ belong to the Boundary $L(u)$, since the ray $\{\lambda x \mid \lambda \geqq 0\}$ coincides with $L(u)$ for $\lambda \geqq \lambda_o$ by virtue of property P.2.

2. PROPERTIES OF THE DISTANCE FUNCTION $\Psi(u,x)$

The properties of $\Psi(u,x)$ are given by the following proposition:

Proposition 3: If the production possibility sets $L(u)$ have the properties P.1, ... P.8, then for any $u > 0$

D.1 $\Psi(u,x) = 0$ for all $x \in \{0\} \cup D_2''$.

D.2 $\Psi(u,x)$ is finite for finite $x \in D$ and positive for all $x \in D_1 \cup D_2'$.

D.3 $\Psi(u,\lambda x) = \lambda\Psi(u,x)$ for $\lambda \geqq 0$ and all $x \in D$.

D.4 $\Psi(u,x+y) \geqq \Psi(u,x) + \Psi(u,y)$ for all $x , y \in D$.

D.5 $\Psi(u,x') \geqq \Psi(u,x)$ if $x' \geqq x \in D$.

D.6 $\Psi(u,x)$ is a concave function of x on D .

D.7 $\Psi(u,x)$ is a continuous function of x on D .

D.8 For any $x \in D$, $\Psi(u_2,x) \leqq \Psi(u_1,x)$ if $u_2 \geqq u_1 > 0$.

D.9 For any $x \in D$, $\inf_{u\to\infty} \Psi(u,x) = 0$.

D.10 If there exists for $\delta > 0$ an open neighborhood $N(0) = \left\{x \,\middle|\, ||x|| < \delta , x \in D\right\}$ such that $x \notin L(u)$ for any $u > 0$ when $x \in N(0)$, then $\sup_{u\to 0} \Psi(u,x)$ is bounded for all $x \in D$, otherwise $\Psi(u,x)$ is surely bounded only for $x \in \{0\} \cup D_2''$ as $u \to 0$.

D.11 For any $x \in D$, $\Psi(u,x)$ is an upper semi-continuous function of u for all $u \in (0,\infty)$.

Property D.1 is merely a restatement of the second part of the definition 1. For $x \in D_1 \cup D_2'$, $||x|| > 0$ and finite for finite x : also $||\xi|| > 0$, since by P.1 $\xi \neq 0$ for $u > 0$. Hence $\Psi(u,x)$ is finite for finite x and positive for $x \in D_1 \cup D_2'$.

Property D.3 holds when $x \in D_1 \cup D_2'$ and $\lambda \geqq 0$, because

$$\Psi(u,\lambda x) = \frac{\|\lambda x\|}{\|\xi\|} = \frac{\lambda\|x\|}{\|\xi\|} = \lambda\Psi(u,x) \ ,$$

since the intersection ξ is fixed for all $\lambda > 0$ and $\Psi(u,0) = 0$; if $x \in \{0\} \cup D_2''$, $\Psi(u,\lambda x) = \lambda\Psi(u,x) = 0$ for all $\lambda > 0$, because by the definition of D_2'' it follows that $(\lambda x) \in \{0\} \cup D_2''$ when $x \in \{0\} \cup D_2''$. Thus $\Psi(u,x)$ is linear homogeneous and D.3 holds for all $x \in D$.

For the verification of property D.4, note first that if $x , y \in D_1 \cup D_2'$ then by D.2 $\Psi(u,x) > 0$, $\Psi(u,y) > 0$ and by D.3

$$\Psi\left(u \ , \ \frac{x}{\Psi(u,x)}\right) = \Psi\left(u \ , \ \frac{y}{\Psi(u,y)}\right) = 1 \ .$$

By Proposition 1 the points $\frac{x}{\Psi(u,x)}$, $\frac{y}{\Psi(u,y)}$, belong to $L(u)$, and since $L(u)$ is convex by P.8 it follows that the point

$$\left[(1 - \theta)\ \frac{x}{\Psi(u,x)} + \theta\ \frac{y}{\Psi(u,y)}\right]$$

belongs to $L(u)$ for any scalar θ satisfying $0 \leqq \theta \leqq 1$. Then by Proposition 1 it follows that

$$\Psi\left(u \ , \ (1 - \theta)\ \frac{x}{\Psi(u,x)} + \theta\ \frac{y}{\Psi(u,y)}\right) \geqq 1$$

for all $\theta \in [0,1]$. Take

$$\theta = \frac{\Psi(u,y)}{\Psi(u,x) + \Psi(u,y)}$$

and use property D.3 to obtain

$$\Psi(u \ , \ x + y) \geqq \Psi(u,x) + \Psi(u \ , \ y) \quad \forall \ x , y \in D_1 \cup D_2' \ .$$

When x and y belong to $\{0\} \cup D_2''$, $\Psi(u,x) = \Psi(u,y) = 0$ by D.1 and the inequality still holds since $\Psi(u , x + y) \geqq 0$. Finally, if one point (say y) belongs to $\{0\} \cup D_2''$ and the other (say x) belongs to $D_1 \cup D_2'$, then $\Psi(u,y) = 0$ by D.1 and $\Psi(u,x) > 0$ by D.2. Then by property P.2 if follows that that the point $\frac{x + y}{\Psi(u,x)}$ belongs to $L(u)$ and by Proposition 1 we have

$$\Psi\left(u , \frac{x + y}{\Psi(u,x)}\right) \geqq 1 ,$$

which implies

$$\Psi(u , x + y) \geqq \Psi(u,x) = \Psi(u,x) + \Psi(u,y)$$

due to the homogeneity of $\Psi(u,x)$. Thus the distance function is super-additive on D .

Property D.5 is a simple consequence of D.4 and the nonnegativity of the distance function, because $x' = x + \Delta x$ where $\Delta x = (x' - x) \geqq 0$ and $\Psi(u,x') \geqq \Psi(u,x) + \Psi(u,\Delta x) \geqq \Psi(u,x)$.

The concavity of $\Psi(u,x)$ on D , i.e., property D.6, follows directly from the super-additivity and homogeneity properties merely by taking $x = (1 - \theta)z$, $y = \theta w$ for any $\theta \in [0,1]$ and any $z , w \in D$ to obtain

$$\Psi(u , (1 - \theta)z + \theta w) \geqq (1 - \theta)\Psi(u,z) + \theta\Psi(u,w) .$$

The continuity in x of the distance function on the closed set D (D.7) may be established as follows: First, for any $u > 0$ the function $\Psi(u,x)$ is continuous on the interior of D , i.e., for $x \in D_1$, by virtue of a well known theorem that a convex function defined on a convex open set in R^n is continuous on this open set (see [1], p193), since $(-\Psi(u,x))$ is convex on the open convex set D_1 . Second regarding the boundary of D , i.e., for $x \in \{0\} \cup D_2$, $\Psi(u,x)$

is lower semi-continuous (see Theorem, p31, [3]). But the distance function is also upper semi-continuous for $x \in \{0\} \cup D_2$, because for any $u > 0$ and any value $v \in [0,\infty)$, the set

$$\{x \mid \Psi(u,x) \geqq v \ , \ x \in R^n\} \ , \ u > 0$$

is closed, which is an if and only if condition for the upper semi-continuity of $\Psi(u,x)$ on D (see [1], p76]). The closure of the level sets of $\Psi(u,x)$ for any $u > 0$ is established as follows:

$$\{x \mid \Psi(u,x) \geqq 0 \ , \ x \in R^n\} = L(0) \ ,$$

since $\Psi(u,x) \geqq 0$ for any $u > 0$ and all $x \in D$, and $L(0)$ is closed. For $v > 0$

$$\{x \mid \Psi(u,x) \geqq v \ , \ x \in R^n\} = \left\{\frac{x}{v} \mid \Psi\left(u,\frac{x}{v}\right) \geqq 1 \ , \ \frac{x}{v} \in R^n\right\}$$

due to the homogeneity of the distance function, and letting $y = \frac{x}{v}$ it follows by Proposition 1 that

$$\{y \mid \Psi(u,y) \geqq 1 \ , \ y \in R^n\} = L(u) \ .$$

Since $L(u)$ is closed (p.7), it follows that

$$\{x \mid \Psi(u,x) \geqq v \ , \ x \in R^n\} \ , \ v > 0$$

is closed. Thus $\Psi(u,x)$ is both lower and upper semi-continuous on $\{0\} \cup D_2$ and therefore continuous on the boundary of D.

Regarding the nonincreasing property (D.8) of $\Psi(u,x)$ in u, for any $x \in D$, suppose first that $x \in D_1 \cup D_2'$ and $u_2 \geqq u_1$. Then by the property P.4 of the sets $L(u)$, $L(u_2) \subset L(u_1)$, and letting ξ_1 and ξ_2 denote the intersections of the ray $\{\lambda x \mid \lambda \geqq 0\}$ with $L(u_1)$ and $L(u_2)$ respectively, it

follows that $\xi_1 \leqq \xi_2$, $||\xi_1|| \leqq ||\xi_2||$ and $\Psi(u_2,x) \leqq \Psi(u_1,x)$. If $x \in \{0\} \cup D_2''$, then by D.1 $\Psi(u_2,x) = \Psi(u_1 x) = 0$ and the inequality is still satisfied.

For the property D.9 note first that $\Psi(u,x) = 0$ for all $u > 0$ when $x \in \{0\} \cup D_2''$. If $x \in D_1 \cup D_2'$, the intersection $\xi(u)$ for any u lies on the ray $\{\lambda x \mid \lambda \geqq 0\}$, and $||\xi(u)||$ cannot be bounded as $u \to \infty$ otherwise $\xi(u) \to \xi_o$ finite, since the sequence $\{\xi(u)\}$ is nondecreasing and bounded above, which implies that there exists a finite point $\xi_o \in D$ which belongs to all level sets $L(u)$ for $u \in [0,\infty)$ by P.4, contrary to P.6. Consequently, $\xi(u) \to \infty$ as $u \to \infty$ for any $x \in D_1 \cup D_2'$ and by the definition (1) it follows that $\Psi(u,x) \to 0$ as $u \to \infty$. Hence $\inf_{u\to\infty} \Psi(u,x) = 0$ for any $x \in D$.

Considering now property D.10, as $u \to 0$ the intersection $\xi(u)$ is non-increasing when $x \in D_1 \cup D_2'$, and unless there exists for $\delta > 0$ an open neighborhood $N(0) = \{x \mid ||x|| < \delta , x \in D\}$ such that $x \notin L(u)$ for any $u > 0$ when $x \in N(o)$, $\{\xi(u)\}$ may $\to 0$ with $\Psi(u,x) \to \infty$ as $u \to 0$. When $x \notin L(u)$ for all $u > 0$ if $x \in N(0)$, $\sup_{u\to 0} \Psi(u,x)$ is bounded, since $\inf_{u\to 0} \xi(u) > 0$. Also $\Psi(u,x) = 0$ for all $u > 0$ when $x \in \{0\} \cup D_2''$ and $\sup_{u\to 0} \Psi(u,x)$ is evidently bounded in this case. Thus D.10 holds.

Regarding property D.11, note first that $\psi(u,x) = 0$ for all $u \in (0,\infty)$ if $x \in \{0\} \cup D_2'$ (see definition (1)), and the distance function is evidently continuous in this case. But when $x \in D_1 \cup D_2'$ a different situation arises. By counter example it may be seen that the distance function is not always lower semi-continuous. Consider the following example: $\Phi(x)$ is a nondecreasing step fucntion, where x is a vector of dimension one, i.e., $x \in R$, as illustrated in Figure 2. This production function is upper semi-continuous and satisfies all of the assumptions A.1, ... A.6. The corresponding distance function is only upper semi-continuous as shown in Figure 3. For any $(i < u \leqq i + 1)$ where

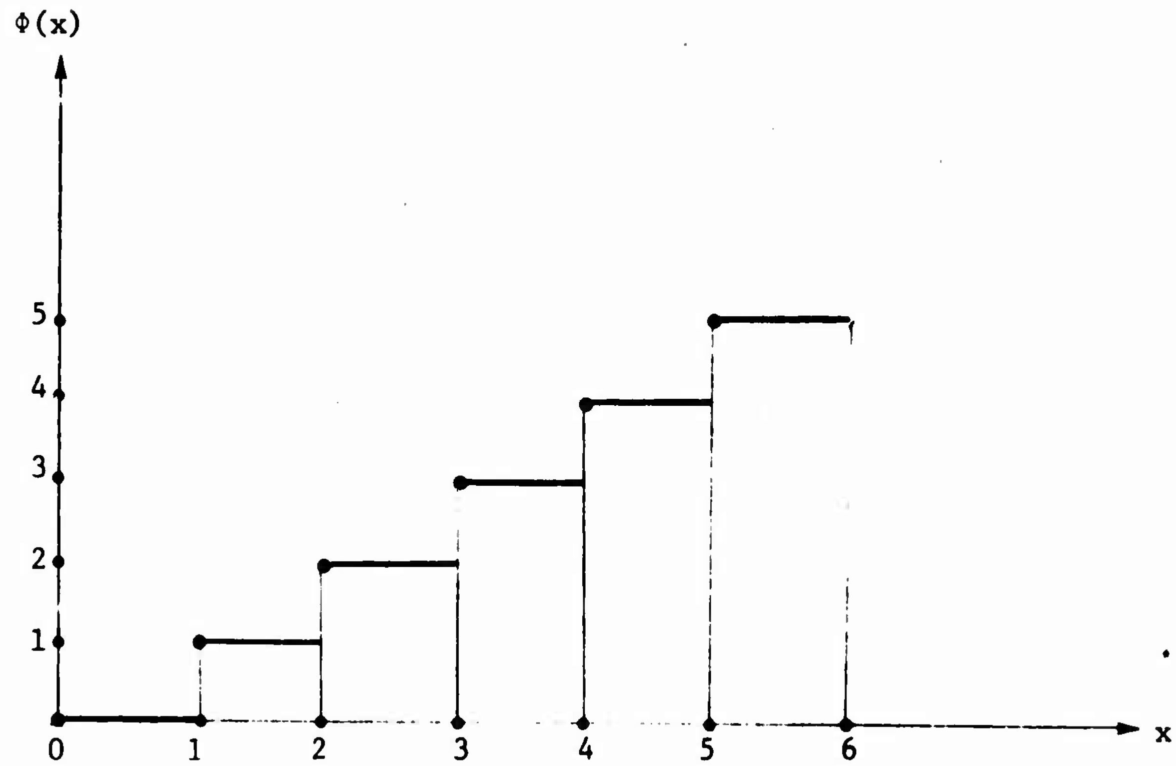

FIGURE 2: UPPER SEMI-CONTINUOUS PRODUCTION STEP FUNCTION

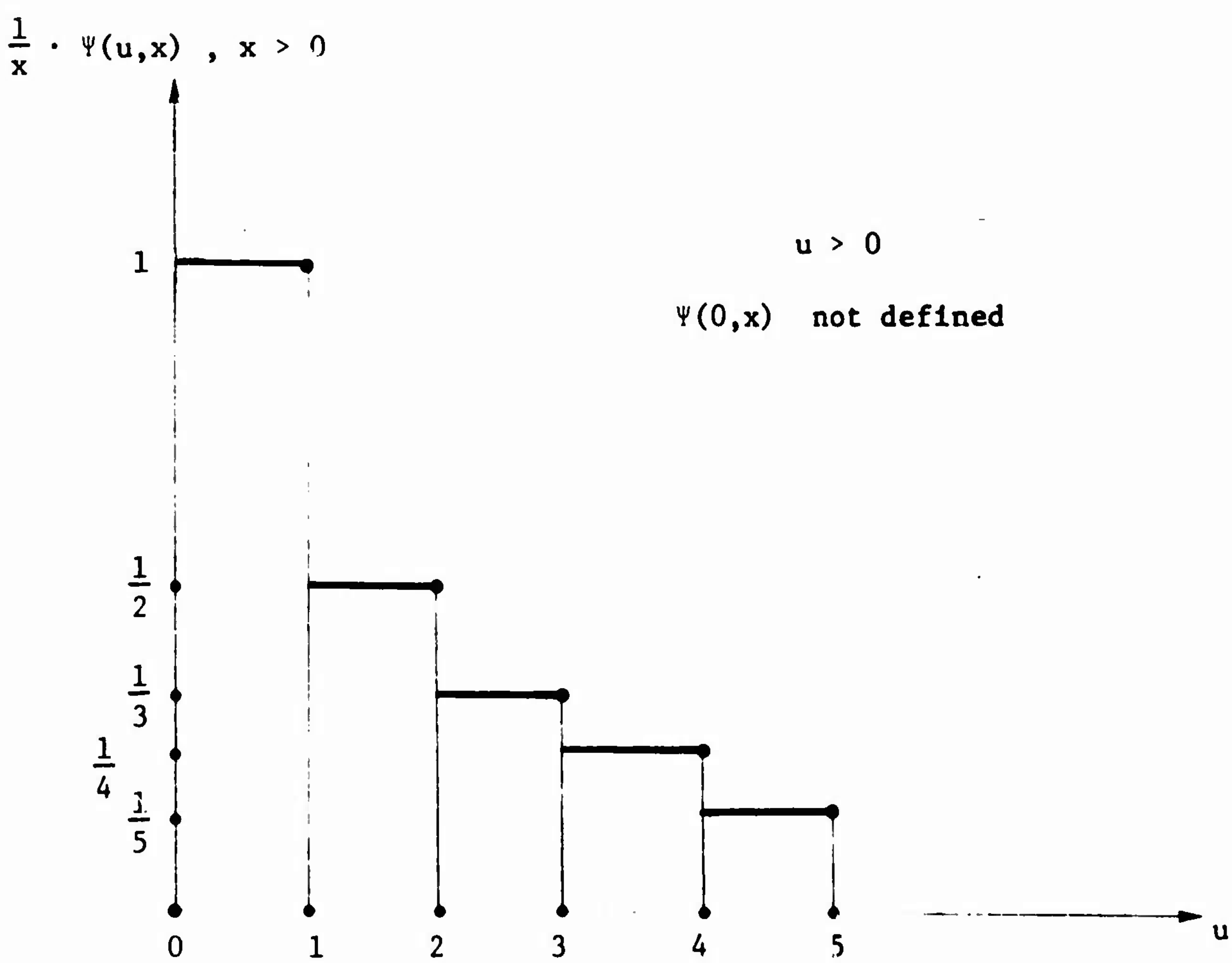

FIGURE 3: DISTANCE FUNCTION FOR THE PRODUCTION STEP FUNCTION

$i \in \{0, 1, 2, 3, \ldots\}$, $L(u) = \{x \mid x \geqq (i+1)\}$. Hence the corresponding intersection ξ is $(i+1)$ and

$$\frac{1}{x}\,\Psi(u,x) = \frac{1}{\xi} = \frac{1}{(i+1)} .$$

The function $\frac{1}{x}\,\Psi(u,x)$ is clearly not lower semi-continuous because let $u = 2$, for example. Then, for any $u > 2$, no matter how close to $u = 2$, $\frac{1}{x}\,\Psi(u,x) < \frac{1}{x}\,\Psi(2,x) - \alpha$ or $\Psi(u,1) < \Psi(2,1) - \alpha$ for $0 < \alpha < 1$ and $u > 2$. However, the distance function is upper semi-continuous for all $x \in D_1 \cup D_2'$ and we proceed to verify this statement.

Let x be any point belonging to $D_1 \cup D_2'$, and consider an arbitrary value of $u \in (0,\infty)$, say u_o . Corresponding to u_o ,

$$\Psi(u_o,x) = \frac{||x||}{||\xi^o||}$$

where $\xi_o = \lambda_o x$ and $\lambda_o = \text{Min}\,\lambda$ for $\lambda x \in L(u_o)$ (see (2)). For all $u \geqq u_o$, $\Psi(u,x) \leqq \Psi(u_o,x)$ (see property D.8) and for any $\alpha > 0$, $\Psi(u,x) < \Psi(u_o,x) + \alpha$ if $u \geqq u_o$. Hence, to show the upper semi-continuity of $\Psi(u,x)$ we need concern ourselves only with values $u < u_o$. Now for all scalars λ such that

$$\frac{\lambda_o}{\alpha\lambda_o + 1} < \lambda < \lambda_o$$

we have

$$\left(\frac{\lambda_o}{\alpha\lambda_o + 1}\right) x < \lambda x < \lambda_o x$$

and

$$\frac{\xi_o}{1 + \lambda_o \alpha} < \xi < \xi_o$$

where ξ is a point on the ray $\{\lambda x \mid \lambda \geqq 0\}$. For such points ξ it is clear for any $\alpha > 0$ that

$$\Psi(u,x) = \left\|\frac{x}{\xi}\right\| < \left\|\frac{x}{\xi_o}\right\|(1 + \lambda_o\alpha) = \left\|\frac{x}{\xi_o}\right\| + \alpha = \Psi(u_o,x) + \alpha .$$

Let $\bar{u} = \text{Max } u$ for $\dfrac{\xi_o}{1 + \lambda_o\alpha} \varepsilon L(u)$ and $\bar{u} < u_o$, since $\xi^o = \lambda_o x$ where $\lambda_o = \text{Min } \lambda$ for $\lambda x \varepsilon L(u_o)$. Then, for all $u \varepsilon (\bar{u},u_o]$ and $u \geqq u_o$, $\Psi(u,x) < \Psi(u_o,x) + \alpha$ for any $\alpha > 0$, and the distance function is upper semi-continuous.

3. EXPRESSION OF THE PRODUCTION FUNCTION IN TERMS OF THE DISTANCE FUNCTION $\Psi(u,x)$

It is clear from Figure 2 and the discussion related thereto that the largest output rate related to an input vector x^o cannot be determined merely from the equation

$$\Psi(u,x) = 1 , \tag{4}$$

since the output rate is not always uniquely determined by this relation. In fact, if the production function $\Phi(x)$ is discontinuous at a point x^o (as illustrated in Figure 4) equation (4) is satisfied by all $u \varepsilon [u_1, u_o]$ where $u_1 = \operatorname*{Sup}_{\lambda<1} \Phi(\lambda x^o)$ and $u_o = \operatorname*{Max}_{x^o \varepsilon L(u)} u$.

The connection between the production function and the distance function is evidently given by

$$\Phi(x) = \begin{cases} \operatorname*{Max}_{\Psi(u,x) \geqq 1} u & \text{for any } x \varepsilon D_1 \cup D_2' \\ \\ 0 & \text{for any } x \varepsilon \{0\} \cup D_2'' . \end{cases} \tag{5}$$

The Max u in (5) exists for $x \varepsilon D_1 \cup D_2'$, because $x \varepsilon L(u)$ if and only if $\Psi(u,x) \geqq 1$ (see Proposition 1) and $\operatorname*{Max}_{x \varepsilon L(u)} u$ has been shown to exist (see [3]). If $x \varepsilon \{0\} \cup D_2''$, then from (1) $\Psi(u,x) = 0$ for all $u > 0$ and the inequality $\Psi(u,x) \geqq 1$ cannot be satisfied by any positive output rate u . However, when $x = 0$ then $\Phi(x) = 0$ (see property A.1) and when $x \varepsilon D_2''$ we have by the definition of D_2'' that $(\lambda x) \notin L(u)$ for any $u > 0$, $\lambda > 0$. Thus in (5) the formula $\Phi(x) = 0$ for $x \varepsilon \{0\} \cup D_2''$ is valid.

Hence the definition of the production function given by (5) is equivalent to $\operatorname*{Max}_{x \varepsilon L(u)} u$ and the function so defined satisfies the properties A.1 , . . . A.6 (see [3]). Accordingly the distance function may be used to define the technological alternatives in production, since the production possibility sets L(u) may

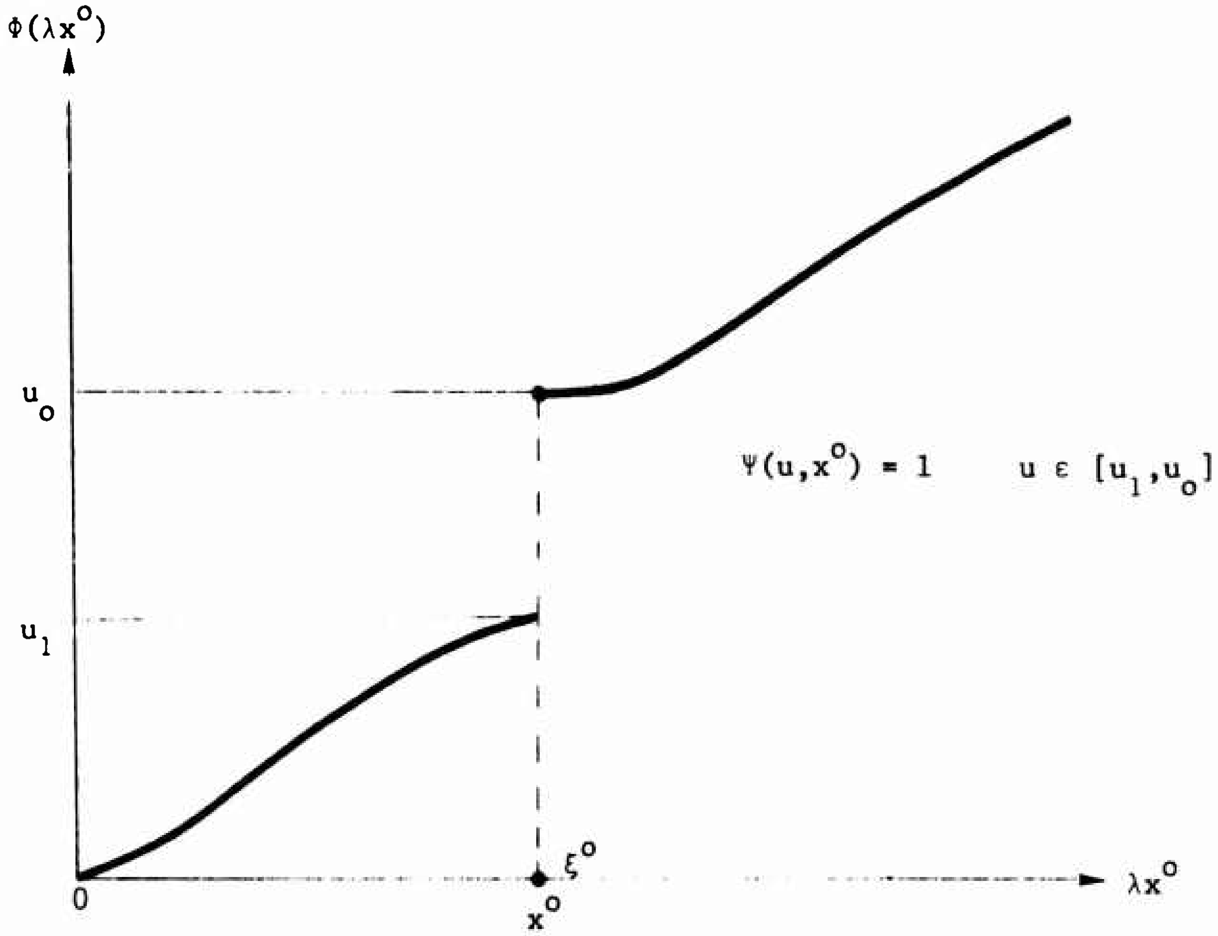

FIGURE 4: NONUNIQUENESS IN u OF DISTANCE FUNCTION

be determined from it by (3) and the maximum output obtainable for any input vector $x \in D$ is given in terms of $\Psi(u,x)$ by (5).

4. THE DISTANCE FUNCTION OF A HOMOTHETIC PRODUCTION STRUCTURE

As defined in [3], a homothetic production structure is one with a production function of the form $F(\Phi(x))$ where $\Phi(x)$ is a homogeneous function satisfying A.1, ... A.6 and $F(\cdot)$ is any nonnegative, continuous, strictly increasing function with $F(0) = 0$ and $F(v) \to \infty$ as $v \to \infty$. As shown in [3] (see Proposition 6), the homothetic production function $F(\Phi(x))$ is continuous for $x \varepsilon D$.

The production possibility sets (level sets) of the homothetic production structure are

$$\begin{aligned} L_F(u) &= \{x \mid F(\Phi(x)) \geqq u \ , \ x \varepsilon D\} \\ &= \{x \mid \Phi(x) \geqq f(u) \ , \ x \varepsilon D\} = L_\Phi(f(u)) \end{aligned}$$

where $f(u)$ is the inverse function of $F(\cdot)$. For the positive output rate u , the level sets may be defined by

$$L_\Phi(f(u)) = \left\{x \;\middle|\; \frac{\Phi(x)}{f(u)} \geqq 1 \ , \ x \varepsilon D\right\}$$

which suggests (see Proposition 1) that the distance function of a homothetic production structure is given by

$$\Psi(u,x) = \frac{\Phi(x)}{f(u)} \ , \ u > 0 \ . \tag{6}$$

If $x \varepsilon \{0\} \cup D_2''$, then $\Phi(0) = 0$ since $\Phi(x)$ has the property A.1 and $\Phi(x) = 0$ for $x \varepsilon D_2''$ by definition of the set D_2'' , while $f(u) > 0$ for $u > 0$. Thus the expression (6) is valid when $x \varepsilon \{0\} \cup D_2''$.

If $x \varepsilon D_1 \cup D_2'$ and $u > 0$, let $\xi = \lambda_o x$ denote the point on the ray $\{\lambda x \mid \lambda \geqq 0\}$ where λ_o equals the Min λ for $\lambda x \varepsilon L(u)$. The point $\xi = \lambda_o x$ is a boundary point of the level set $L_\Phi(f(u))$ and, since $\Phi(x)$ is continous and strictly increasing along the ray $\{\lambda x \mid \lambda \geqq 0\}$, it follows that

$\Phi(\xi) = \lambda_o \Phi(x) = f(u)$. Hence

$$\lambda_o = \frac{f(u)}{\Phi(x)} , \ \Phi(x) > 0 .$$

Then, using the definition (1) for any $x \varepsilon D_1 \cup D_2'$

$$\Psi(u,x) = \frac{||x||}{||\lambda_o x||} = \frac{1}{\lambda_o} = \frac{\Phi(x)}{f(u)}$$

and (6) is a proper formula for the distance function of a homothetic production structure.

That the expression (6) satisfies some of the properties D.1, ... D.11 is more or less evident. Property D.1 has been shown above in verifying the equivalence for (6) with (1). For x finite, $\Phi(x)$ satisfies A.2 and is finite with $f(u) > 0$, and also $\Phi(x) > 0$ for $x \varepsilon D_1 \cup D_2'$ since $\Phi(\lambda x) > 0$ for some $\lambda > 0$ and the homogeneity of $\Phi(\lambda x)$ implies $\Phi(x) > 0$. Thus D.2 holds. Property D.3 holds, since

$$\Psi(u,\lambda x) = \frac{\Phi(\lambda x)}{f(u)} = \frac{\lambda\Phi(x)}{f(u)} = \lambda\Psi(u,x)$$

due to the homogeneity of $\Phi(x)$. The satisfaction of the properties D.4, D.6, and D.7 is less obvious. But since $\Phi(x)$ is nondecreasing in x (A.3), homogeneous, upper semi-continuous (A.5) and quasi-concave (A.6) for all $x \varepsilon D$, it follows from Proposition 6 of [3] that $\Phi(x)$ is a super-additive, concave and continuous function of x for all $x \varepsilon D$. Then property D.6 follows directly from the nonnegativity and super-additivity of $\Phi(x)$. Property D.8 follows directly from the strictly increasing property of $f(u)$, and D.9 holds since $f(u) \to \infty$ as $u \to \infty$. However property D.10 is strengthened to: for any $x \varepsilon D_1 \cup D_2''$ the distance function $\frac{\Phi(x)}{f(u)} \to \infty$ as $u \to 0$ since $f(u) \to 0$ as $u \to 0$, while for $x \varepsilon \{0\} \cup D_2''$, $\Phi(x) = 0$ and $\frac{\Phi(x)}{f(u)} = 0$ for all $u \varepsilon (0,\infty)$. Finally, D.11 is strengthened to $\frac{\Phi(x)}{f(u)}$ is continuous for all $u \varepsilon (0,\infty)$, since $f(u)$ is a continuous function of u .

In general, no simple explicit formula like (6) relates the production function $\Phi(x)$ and the distance function $\Psi(u,x)$, when the production structure is not homothetic. However, for any $x \varepsilon D_1 \cup D_2'$,

$$\Psi\left(u , \frac{x}{\Psi(u,x)}\right) = 1$$

due to the homogeneity of the distance function, and the point $\frac{x}{\Psi(u,x)}$ lies on the boundary of the level set $L(f(u))$, taking $F(\Phi(x))$ as the production function with $\Phi(x)$ satisfying only A.1, ... A.6. On the boundary of $L(f(u))$ we have $\Phi(x) = f(u)$, if u is a realizable output rate. Thus, if the output rate u is "realizable,"* the production function and distance function are related by

$$\Phi\left(\frac{x}{\Psi(u,x)}\right) = f(u) . \tag{7}$$

*i.e., one for which there exists x such that $\Phi(x) = u$.

BIBLIOGRAPHY

[1] Berge, C., Topological Spaces, Macmillan Co., New York, (1963).

[2] Minkowski, H., Theorie der Konvexen Körper, Gesammelte Abhandlungen, Bd. 2, (1911).

[3] Shephard, Ronald W., "On the Notion of a Production Function," Operations Research Center Report ORC 66-42, University of California, Berkeley, (1966).

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) University of California	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

THE DISTANCE FUNCTION OF A PRODUCTION STRUCTURE

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Research Report

5. AUTHOR(S) (Last name, first name, initial)

Shephard, Ronald W.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
January 1967	22	3

8a. CONTRACT OR GRANT NO. Nonr-222(83 b. PROJECT NO. NR 047 033 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) ORC 67-4 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited

11. SUPPLEMENTARY NOTES Also supported by the National Science Foundation	12. SPONSORING MILITARY ACTIVITY Mathematical Science Division Office of Naval Research

13. ABSTRACT

DD FORM 1 JAN 64 1473

Unclassified

Security Classification

Unclassified

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Theory of Production Distance Function Representation of Production Structure						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through ________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through ________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through ________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS), (S), (C), or (U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

DD FORM 1 JAN 64 1473 (BACK)

Unclassified

Security Classification

ADDENDUM TO

THE DISTANCE FUNCTION OF A PRODUCTION STRUCTURE

by

Ronald W. Shephard
ORC 67-4
September 1967

(1) Page 3, definition (1):

Modify to include $\Psi(0, x) = +\infty$ for all $x \in D$, by considering the extended real number system with $+\infty$ adjoined to denote $\sup\limits_{u \geqq 0} u$. For $x \geq 0$ (semi-positive) it is intuitively clear why this extended definition is used and when $x = 0$ the same definition will serve our purposes.

(2) Page 5, Proposition 1:

Write equation (3) as:

$$L(u) = \{x \mid \Psi(u, x) \geqq 1, x \geqq 0\}, u \geqq 0$$

since, with $\Psi^*(0, x) = +\infty$ for all $x \in D$, it is clear that $L(0) = D$.

(3) Page 7, Proposition 3:

Restate to include $u = 0$, by the following alterations:

D.1 $\Psi(0, x) = +\infty \ \forall \ x \in D$, and $\Psi(u, x) = 0$ $\forall u > 0, x \in \{0\} \cup D_2''$.

D.2 For all $u > 0$, $\Psi(u, x)$ is finite for finite $x \in D$ and positive for $x \in D_1 \cup D_2'$.

D.3 $\Psi(u, \lambda x) = \lambda \Psi(u, x) \ \forall \ u \geqq 0, \lambda \geqq 0, x \in D$, but $\lambda = u \neq 0$.

D.4 $\Psi(u, x + y) \geqq \Psi(u, x) + \Psi(u, y) \quad \forall \quad u \geqq 0, x \in D, y \in D$.

D.5 $\Psi(u, x') \geqq \Psi(u, x) \; \forall \; u \geqq 0$, if $x' \geqq x$.

D.6 $\Psi(u, x)$ is a concave function of x on D for all $u \geqq 0$.

D.7 $\Psi(u, x)$ is a continuous function of x on D for all $u \geqq 0$.

D.8 For any $x \in D$, $\Psi(u_2, x) \leqq \Psi(u_1, x)$ if $u_2 \geqq u_1 \geqq 0$.

D.9 For any $x \in D$, $\liminf_{u \to \infty} \psi(u, x) = 0$.

D.10 For any $x \in D$, $\liminf_{u \to 0} \Psi(u, x) \leqq + \infty$ and $< \infty$ is possible.

D.11 For any $x \in D$, $\Psi(u, x)$ is an upper semi-continuous function of u for all $u \in [0, \infty)$.

Property D.10 as stated here is a simplification of statement suitable for our purposes. The possibility of $\liminf_{u \to 0} \Psi(u, x)$ being bounded still applies and is indicated by the inequality sign.

These restatements of the properties evidently hold when $\Psi(0, x) = + \infty$ for all $x \in D$.

(4) Page 15, Equation (5) should be restated as:

$$\Phi(x) = \text{Max} \{u \mid \Psi(u, x) \geqq 1\}, \; x \in D \tag{5}$$

to include the case where $x \in \{0\} \cup D_2''$. This extension is correct, because

$$\{u \mid \Psi(u, x) \geqq 1, \; x \in \{0\} \cup D_2''\} = \{0\}$$

and hence for $x \in \{0\} \cup D_2''$

$$\Phi(x) = \text{Max} \{u \mid \Psi(u, x) \geqq 1\} = 0$$

The alteration of property D.10 suggested at the bottom of Page 19 should be omitted in favor of the revised D.10 stated above, which is clearly satisfied by $\Phi(x)/f(u)$ for all $x \in D$, $u \in [0, \infty)$.